An Identity of A. Ostrowski*

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Let A be an $n \times n$ matrix with complex entries; x and y, $n \times 1$ vectors; and f, the bilinear form $y'Ax$. In [1] A. Ostrowski derives an identity for the determinant of f when certain linear relations exist among the components of x and y, which is a generalization of an identity due to R. Šostak (consult Ostrowski's article for a reference to Šostak's paper). In this note we give another proof of Ostrowski's identity which uses nothing more than partitioning of a matrix.

We denote the unit matrix of order r by I_r.

Let k be a positive integer less than n. Suppose that B and C are $k \times n$ matrices with complex entries such that $Bx = Cy = 0$, $B = [B_0 B_1]$, $C = [C_0 C_1]$, and the $k \times k$ matrices B_0 and C_0 are nonsingular. Put $\det B_0 = b_0$, $\det C_0 = c_0$. Let D be the $(n-k) \times (n-k)$ matrix of the bilinear form f when the first k components of x and y are eliminated, and d its determinant. Let Δ be the $(n+k) \times (n+k)$ matrix

$$\Delta = \begin{bmatrix} A & C' \\ B & 0 \end{bmatrix},$$

and δ its determinant. Then Ostrowski's identity is

$$d = (-1)^k (b_0 c_0)^{-1} \delta. \tag{1}$$

Partition x into $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ and y into $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$, where x_0 and y_0 are $k \times 1$ vectors.

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* Dedicated to Professor A. M. Ostrowski on his 75th birthday.

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Then

$$B_0 x_0 + B_1 x_1 = C_0 y_0 + C_1 y_1 = 0.$$

Set

$$B_2 = -B_0^{-1} B_1, \qquad C_2 = -C_0^{-1} C_1,$$

so that B_2 and C_2 are $k \times (n-k)$ matrices and $x_0 = B_2 x_1$, $y_0 = C_2 y_1$. Partition A into

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1 is a $k \times k$ matrix, A_2 a $k \times (n-k)$ matrix, A_3 an $(n-k) \times k$ matrix, and A_4 an $(n-k) \times (n-k)$ matrix. Then upon elimination of the redundant vectors x_0 and y_0 the bilinear form f becomes

$$\begin{aligned} f = y'Ax &= [y_1'C_2' y_1'] \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_2 x_1 \\ x_1 \end{bmatrix} \\ &= y_1' D x_1, \end{aligned}$$

where

$$D = C_2' A_1 B_2 + A_3 B_2 + C_2' A_2 + A_4. \tag{2}$$

Put

$$L_1 = \begin{bmatrix} I_k & 0 & 0 \\ 0 & I_{n-k} & 0 \\ 0 & 0 & B_0^{-1} \end{bmatrix}, \qquad R_1 = \begin{bmatrix} I_k & 0 & 0 \\ 0 & I_{n-k} & 0 \\ 0 & 0 & (C_0')^{-1} \end{bmatrix}.$$

Then

$$L_1 A R_1 = \begin{bmatrix} A_1 & A_2 & I_k \\ A_3 & A_4 & -C_2' \\ I_k & -B_2 & 0 \end{bmatrix}.$$

Put

$$L_2 = \begin{bmatrix} I_k & 0 & 0 \\ C_2' & I_{n-k} & 0 \\ 0 & 0 & I_k \end{bmatrix}, \qquad R_2 = \begin{bmatrix} I_k & B_2 & 0 \\ 0 & I_{n-k} & 0 \\ 0 & 0 & I_k \end{bmatrix}.$$

Then

$$L_2L_1\Delta R_1R_2 = \begin{bmatrix} A_1 & A_2 + A_1B_2 & I_k \\ A_3 + C_2'A_1 & D & 0 \\ I_k & 0 & 0 \end{bmatrix},$$

in view of (2).

Put

$$R_3 = \begin{bmatrix} 0 & 0 & I_k \\ 0 & I_{n-k} & 0 \\ I_k & 0 & 0 \end{bmatrix}.$$

Then

$$L_2L_1\Delta R_1R_2R_3 = \begin{bmatrix} I_k & A_2 + A_1B_2 & A_1 \\ 0 & D & A_3 + C_2'A_1 \\ 0 & 0 & I_k \end{bmatrix}. \tag{3}$$

It is easily seen that $\det L_1 = b_0^{-1}$, $\det R_1 = c_0^{-1}$, $\det L_2 = \det R_2 = 1$, and $\det R_3 = (-1)^k$. Since the determinant of the right side of (3) is d, it follows that

$$(-1)^k(b_0c_0)^{-1}\delta = d,$$

which completes the proof of (1).

As Ostrowski pointed out, (3) also implies the following: As usual, define the *nullity* of a matrix as the dimension of its null space, which equals the number of its columns less its rank. Let D have rank r. Then D has nullity $n - k - r$. Now the matrix on the right side of (3) has rank $2k + r$ and nullity $n + k - (2k + r) = n - k - r$. Since L_1, R_1, L_2, R_2, R_3 are nonsingular, the same is true of Δ. Thus

$$\text{the nullity of } D \text{ equals the nullity of } \Delta. \tag{4}$$

REFERENCE

1 A. Ostrowski, Über geänderte Determinanten und bedingte Trägheitsindizes quadratischer Formen. *Monatsh. Math.* **64**(1960), 51–63.

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